

Neural Codes and Neural Ideals

A **neural code** is a subset $\mathcal{C} \subseteq \{0, 1\}^n$, which we interpret as a set of binary patterns of neural activity for a set of neurons labeled $[n] = \{1, \dots, n\}$.

A **pseudomonomial** is a polynomial of the form

$$f = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j) \in R := \mathbb{F}_2[x_1, \dots, x_n]$$

for some $\sigma, \tau \subseteq [n]$ with $\sigma \cap \tau = \emptyset$.

Given a neural code $\mathcal{C}$, the associated **neural ideal** is

$$J_{\mathcal{C}} = \left(f \text{ a pseudomonomial} \mid f(c) = 0 \text{ for all } c \in \mathcal{C} \right) \subseteq R.$$

Example: The set $\mathcal{C} = \{000, 100, 010, 101\}$ is a neural code on 3 neurons and the corresponding neural ideal is $J_{\mathcal{C}} = (x_3(1 - x_1), x_1x_2)$. Notice that

$$x_2(x_3(1 - x_1)) + x_3(x_1x_2) = x_2x_3 \in J_{\mathcal{C}}.$$

Polarization

Let $S = R[y_1, \dots, y_n]$. The following map of pseudomonomials

$$\mathcal{P} : R \longrightarrow S \\ \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j) \longmapsto \prod_{i \in \sigma} x_i \prod_{j \in \tau} y_j$$

is called the **polarization map**. [2]

Given a neural code $\mathcal{C}$, the **polarization of $J_{\mathcal{C}}$** is

$$\mathcal{P}(J_{\mathcal{C}}) := \left(\mathcal{P}(f) \mid f \in J_{\mathcal{C}} \text{ is a pseudomonomial} \right) \subseteq S.$$

Example: Recall the neural ideal $J_{\mathcal{C}} = (x_3(1 - x_1), x_1x_2) \subseteq R$. If we polarize these generators, we get

$$I = (x_3y_1, x_1x_2) \neq \mathcal{P}(J_{\mathcal{C}})$$

because $x_2x_3 \in J_{\mathcal{C}}$ and $\mathcal{P}(x_2x_3) \notin I$. The polarization of $J_{\mathcal{C}}$ is actually

$$\mathcal{P}(J_{\mathcal{C}}) = (x_3y_1, x_1x_2, x_2x_3) \subseteq S.$$

We say that a squarefree monomial ideal $I \subseteq S$ is a **polarized neural ideal (PNI)** if there exists a neural code $\mathcal{C}$ such that $I = \mathcal{P}(J_{\mathcal{C}})$.

Example: $\mathcal{P}(J_{\mathcal{C}}) = (x_3y_1, x_1x_2, x_2x_3)$ is a PNI, $I = (x_3y_1, x_1x_2)$ is not a PNI.

A monomial m is **non-Boolean** if $x_i y_i \nmid m$ for all i . A squarefree monomial ideal I is non-Boolean if all its minimal monomial generators are non-Boolean.

Remark: If $I \subseteq S$ is a PNI, then I is non-Boolean.

Motivating Question

Question 1: Given a squarefree monomial ideal in $T := \mathbb{F}_2[v_1, \dots, v_{2n}]$ with $2n$ variables, how can we tell if there is a way to relabel the variables so that the ideal is a PNI?

One strategy to answer this question is to study invariants of PNIs.

Question 2: If I is a PNI, what are the possible values for $\mu(I)$?

We will answer this question in the quadratic case, which allows us to conclude that there is **no relabeling** under which

$$I = (v_1v_2, v_3v_4, v_5v_6, v_2v_4, v_3v_5) \subseteq \mathbb{F}_2[v_1, \dots, v_6]$$

is a PNI.

General Bounds

Theorem (Sperner, 1928): Let $I \subseteq S$ be a squarefree monomial ideal. Then

$$\mu(I) \leq \binom{2n}{n}.$$

Equality holds if and only if $I = (m \text{ squarefree monomial} \mid \deg(m) = n)$.

Theorem: Let $n \geq 2$ and let $d = \lceil \frac{2n}{3} \rceil$. If $I \subseteq S$ is a non-Boolean ideal, then

$$\mu(I) \leq \binom{n}{d} 2^d.$$

Equality holds if $I = (m \text{ non-Boolean squarefree monomial} \mid \deg(m) = d)$.

Remark: The maximizers in both of these theorems are not PNIs.

Bounds for Equigenerated Ideals

Consider the following set:

$$\mathcal{V}_n^{(d)} := \{\mu(I) \mid I \subseteq S \text{ is a } d\text{-equigenerated PNI}\}.$$

Theorem: For $1 \leq d \leq n$,

$$\max(\mathcal{V}_n^{(d)}) \leq \binom{n}{d} 2^{d-1}.$$

Equality holds for $d \in \{1, 2, n\}$. Moreover,

$$\mathcal{V}_n^{(1)} = [n] \quad \text{and} \quad \mathcal{V}_n^{(n)} = [2^{n-1}].$$

Quadratic Machinery

To describe $\mathcal{V}_n^{(2)}$, we view a quadratic PNI as the edge ideal of an underlying simple graph. Since PNIs are non-Boolean, these graphs are n -partite via the vertex subsets $Z_i = \{x_i, y_i\}$, which we call the **columns** of the graph.

A **matching** in a bipartite graph is a collection of edges where no two edges share a vertex. A **perfect matching** is a matching where every vertex in the graph is included in some edge from the matching.

Lemma:

- If $I \subseteq S$ is a PNI, then the edges between any two columns in its underlying graph form a matching.
- If $I \subseteq S$ is a PNI, then having a perfect matching between a pair of columns in the underlying graph is an equivalence relation on the columns.
- If $I \subseteq S$ is a PNI and $F, H \subseteq \{Z_1, \dots, Z_n\}$ are equivalence classes under the perfect matching relation, then the number of edges between F and H is either 0 or $|F| \cdot |H|$.

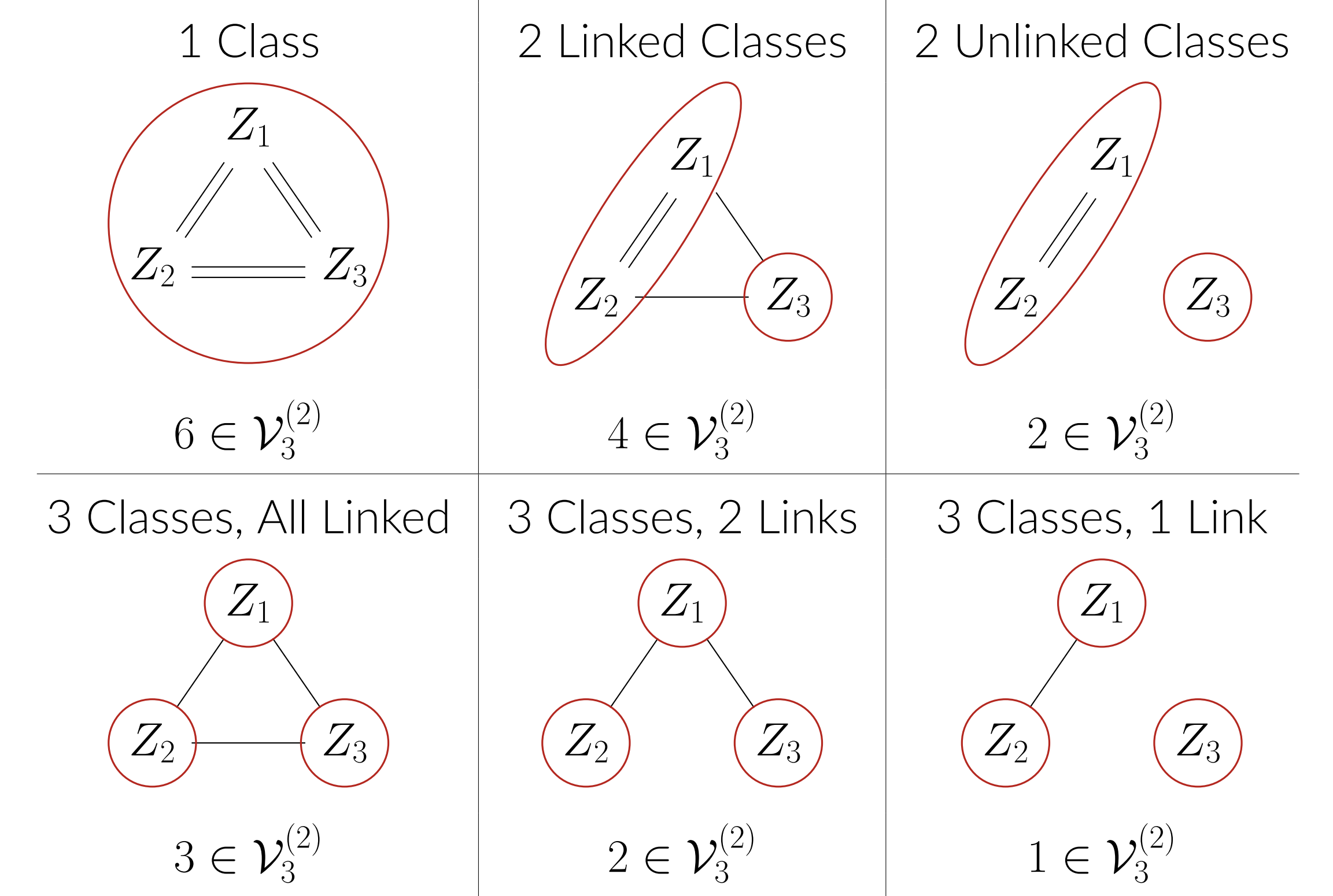
Characterizing $\mathcal{V}_n^{(2)}$

Theorem: Let $n \geq 2$. Then

$$\mathcal{V}_n^{(2)} = \bigcup_{(a_1, \dots, a_k) \vdash n} \left\{ 2 \sum_{i=1}^k \binom{a_i}{2} + \sum_{1 \leq s < t \leq k} \ell_{st} a_s a_t \mid \ell_{st} \in \{0, 1\} \right\} \setminus \{0\}.$$

Example: $\mathcal{V}_3^{(2)}$

To build a quadratic PNI, we must first partition the columns into equivalence classes, then choose which classes are linked.



So $\mathcal{V}_3^{(2)} = \{1, \dots, 4\} \cup \{6\}$.

Quadratic Maximizers

Definition: The **support** of a codeword $c \in \mathcal{C}$ is $\text{supp}(c) = \{i \in [n] \mid c_i = 1\}$.

Theorem: Let $I \subseteq S$ be a quadratic PNI. The following are equivalent:

- $\mu(I) = n(n - 1) = \max(\mathcal{V}_n^{(2)})$.
- For some $\sigma \subseteq [n]$,

$$I = (x_i x_j, y_i y_j \mid \text{exactly one of } i, j \text{ is in } \sigma) \\ + (x_i y_j, y_i x_j \mid \text{none or both of } i, j \text{ is in } \sigma).$$
- $I = \mathcal{P}(J_{\mathcal{C}})$ where $\mathcal{C} = \{c, d\}$ such that $\text{supp}(c) \sqcup \text{supp}(d) = [n]$

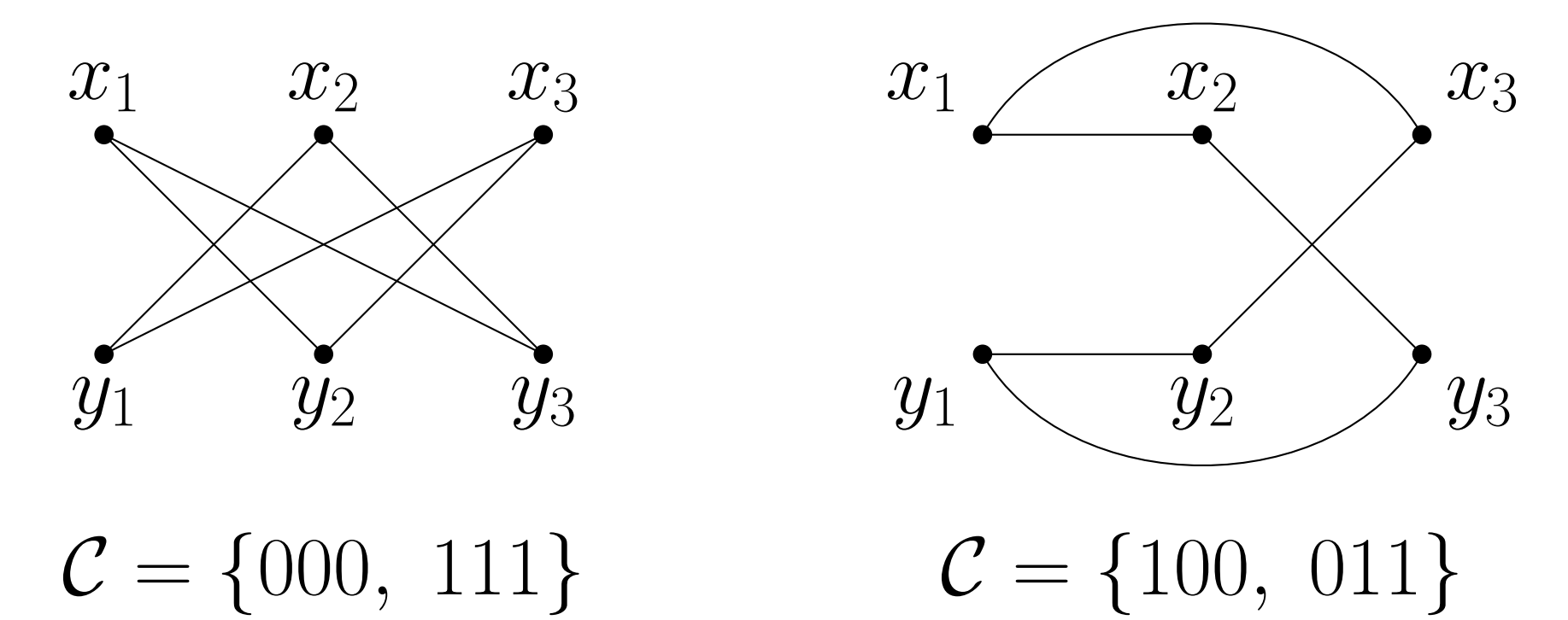
When the equivalent conditions hold,

$$\sigma = \{1\} \cup \{i \mid x_1 y_i \in G(I)\} = \text{supp}(c)$$

in 2. and 3.

Example: Quadratic Maximizers on 3 Neurons

When $n = 3$, there are two maximal quadratic PNIs, up to relabeling the neurons:



References

- [1] Hugh Geller and Rebecca R. G. Canonical forms of neural ideals. *La Matematica*, 3(2):721–752, 2024.
- [2] Sema Gunturkun, Jack Jeffries, and Jeffrey Sun. Polarization of neural rings, 2017.
- [3] Nora Youngs. The neural ring: using algebraic geometry to analyze neural codes, 2014.